Faster Lagrangian-Based Methods in Convex Optimization

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Joint work with

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Optimization without Borders
Dedicated to Yurii Nesterov's 65th Birthday

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HAPPY BIRTHDAY Yurii !

Goal/Outline

Main goal. **To unify, simplify, and improve the convergence rate analysis of Lagrangian-based methods for convex optimization.**

- **A central tool for Lagrangian methods: Nice Primal Algorithmic Map**
- **A framework of Faster LAGrangian (FLAG) methods**
- **New non-ergodic rate of convergence results in terms of function values and feasibility violation**

The Linearly Constrained Convex Optimization Model

We focus on the **linearly constrained** convex optimization problem defined by

$$\text{(P)} \qquad \min_{x \in \mathbb{R}^n} \{\Psi(x) : \mathcal{A}x = b\},$$

where

- $\Psi : \mathbb{R}^n \to (-\infty, +\infty]$ is proper, lsc and σ-strongly convex with $\sigma \geq 0$.
- $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, and $b \in \mathbb{R}^m$.
- The feasible set of problem (P) is denoted by $\mathcal{F} = \{x \in \mathbb{R}^n : \mathcal{A}x = b\} \neq \emptyset$.

Despite its apparent simplicity, this model is very rich and encompasses most convex optimization models.

Important Particular Instances of Model (P)

- **Linear composite model**

$$\min_{u\in\mathbb{R}^p}\{f(u)+g(Au)\}=\min_{u\in\mathbb{R}^p,v\in\mathbb{R}^q}\{f(u)+g(v):\ Au=v\},$$

where $f:\mathbb{R}^p\to(-\infty,+\infty]$ and $g:\mathbb{R}^q\to(-\infty,+\infty]$ are proper, lower semi-continuous and convex functions, and $A:\mathbb{R}^p\to\mathbb{R}^q$ is a linear mapping. **It fits into model (P)**, with $x=\left(u^T,v^T\right)^T$, $\Psi(x):=f(u)+g(v)$ and $\mathcal{A}x=Au-v$.

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- **Block linear constrained model**

$$\min_{u\in\mathbb{R}^p,v\in\mathbb{R}^q}\{f(u)+g(v):\ Au+Bv=b\},$$

where $f:\mathbb{R}^p\to(-\infty,+\infty]$ and $g:\mathbb{R}^q\to(-\infty,+\infty]$ are proper, lower semi-continuous and convex functions, $A:\mathbb{R}^p\to\mathbb{R}^m$ and $B:\mathbb{R}^q\to\mathbb{R}^m$ are linear mappings. **It fits into model (P)**, with $x=\left(u^T,v^T\right)^T$, $\Psi(x):=f(u)+g(v)$ and $\mathcal{A}x=Au+Bv$.

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- **Additive smooth/non-smooth composite objective**

$$\min_{x\in\mathbb{R}^n}\{f(x)+h(x):\ \mathcal{A}x=b\},$$

where $f:\mathbb{R}^n\to(-\infty,+\infty]$ is a proper, lsc and convex function, while $h:\mathbb{R}^n\to\mathbb{R}$ is a continuously differentiable function with a Lipschitz continuous gradient.

Lagrangians for the Convex Model (P)

We recall problem (P)

$$(\mathrm{P}) \qquad \min_{x \in \mathbb{R}^n} \{\Psi(x) : \mathcal{A}x = b\},$$

The corresponding Lagrangian and augmented Lagrangian, are respectively given by:

$$\mathcal{L}(x, y) = \Psi(x) + \langle y, \mathcal{A}x - b \rangle, \quad y \in \mathbb{R}^m,$$

and, for any $\rho > 0$,

$$\mathcal{L}_\rho(x, y) = \mathcal{L}(x, y) + \frac{\rho}{2} \|\mathcal{A}x - b\|^2.$$

Assumption

The Lagrangian $\mathcal{L}$ has a saddle point, that is, there exists (x^*, y^*) such that

$$\mathcal{L}(x^*, y) \leq \mathcal{L}(x^*, y^*) \leq \mathcal{L}(x, y^*), \quad \forall\, x \in \mathbb{R}^n, \quad \forall\, y \in \mathbb{R}^m.$$

It can be warranted, for instance, **under standard CQ on the problem's data**.

Lagrangian-based Methods for Model (P)

Starting point: all Lagrangian-based methods update a couple (x, y) via

$$x^+ \in \mathcal{P}(x, y),$$
$$y^+ = y + \mu\rho\left(\mathcal{A}x^+ - b\right),$$

where $\mathcal{P}(\cdot, \cdot)$ is a primal algorithmic map and $\mu > 0$ is a scaling parameter.

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- The main difference between Lagrangian-based methods is encapsulated through the choice of a primal algorithmic map that updates the primal variable.

- This primal map can be seen as the step of any optimization method that is applied on the augmented Lagrangian itself or a variation of it.

Examples – Some Classical Lagrangian Based Schemes

$$\mathcal{L}_\rho(x,y) = \Psi(x) + \langle y, \mathcal{A}x - b\rangle + \frac{\rho}{2}\|\mathcal{A}x - b\|^2 .$$

Augmented Lagrangian (Hestenes (69), Powell (69))
Main step: Given (x,y), update the new point $\left(x^+, y^+\right)$ via:

$$x^+ \in \operatorname{argmin}\left\{\mathcal{L}_\rho(\xi, y) : \xi \in \mathbb{R}^n\right\},$$
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Proximal Linearized Augmented Lagrangian
Main step: Given (x,y), update the new point (x^+,y^+) via:

$$x^+ \in \operatorname{argmin}\left\{\Psi(\xi) + \left\langle \xi, \mathcal{A}^T\left(y + \rho\left(\mathcal{A}x - b\right)\right)\right\rangle + \frac{1}{2}\|\xi - x\|_M^2 : \xi \in \mathbb{R}^n\right\}, \ (M \succ 0)$$
$$y^+ = y + \mu\rho\left(\mathcal{A}x^+ - b\right).$$

In this case, $\mathcal{P}$ is a **proximal gradient** applied on the augmented Lagrangian.

Examples: Some Classical Schemes for Block Models

As discussed above, Model (P) covers the following block model

$$\min_{(u,v)\in\mathbb{R}^n} \{f(u) + g(v) \ : \ Au + Bv = b\}.$$

$$\mathcal{L}_\rho(u, v, y) = f(u) + g(v) + \langle y, Au + Bv - b\rangle + \frac{\rho}{2}\|Au + Bv - b\|^2.$$

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Alternating Direction Method of Multipliers (ADMM)

(Glowinski and Marroco (75), Gabay and Mercier (76))

Main step: Given (u,v,y), update the new point $\left(u^+,v^+,y^+\right)$ via:

$$\begin{aligned}
u^+ &= \operatorname{argmin}\left\{\mathcal{L}_\rho(\xi,v,y) \,:\, \xi\in\mathbb{R}^n\right\},\\
v^+ &= \operatorname{argmin}\left\{\mathcal{L}_\rho\left(u^+,\eta,y\right) \,:\, \eta\in\mathbb{R}^m\right\},\\
y^+ &= y+\mu\rho\left(Au^+ + Bv^+ - b\right).
\end{aligned}$$

In this case, $\mathcal{P}$ is an **alternating minimization** applied on the augmented Lagrangian.

Examples – Some Classical Schemes for Block Models

$$\mathcal{L}_\rho(u, v, y) = f(u) + g(v) + \langle y, Au + Bv - b \rangle + \frac{\rho}{2} \|Au + Bv - b\|^2 .$$

Previous steps can be difficult to implement. Instead, we can *approximate them*:

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Proximal Linearized ADMM
Main step: Given (u,v,y), update the new point (u^+,v^+,y^+) via:

$$u^+ = \operatorname{argmin}_\xi \left\{ f(\xi) + \left\langle A^T\left(y + \rho\left(Au + Bv - b\right)\right), \xi - u\right\rangle + \frac{1}{2}\|\xi - u\|^2_{M_1} \right\},$$

$$v^+ = \operatorname{argmin}_\eta \left\{ g(\eta) + \left\langle B^T\left(y + \rho\left(Au^+ + Bv - b\right)\right), \eta - v\right\rangle + \frac{1}{2}\|\eta - v\|^2_{M_2} \right\},$$

$$y^+ = y + \mu\rho\left(Au^+ + Bv^+ - b\right).$$

(Here $M_1, M_2 \succ 0$).

In this case, $\mathcal{P}$ is a **alternating proximal gradient** applied on the augmented Lagrangian.

Nice Primal Algorithmic Map

It captures the essential ingredients and plays a central role in unifying the analysis of all Lagrangian-based methods into a single and simple framework.

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Definition (Nice primal algorithmic map)

Given the parameters $\rho, t > 0$, let

$$(\rho_t, \tau_t) := \begin{cases} (\rho, t^{-1}) & \text{if } \sigma = 0 \\ (\rho t, \ \ t) & \text{if } \sigma > 0. \end{cases}$$

A primal algorithmic map $\mathrm{Prim}_t : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ which is applied on the augmented Lagrangian $\mathcal{L}_{\rho_t}(z, \lambda)$ that generates $z^+ \in \mathrm{Prim}_t(z, \lambda)$, is called **nice**, **if there exist** $\delta \in (0, 1]$ **and** $P, Q \succeq 0$, such that for any $\xi \in \mathcal{F}$ we have

$$\mathcal{L}_{\rho_t}(z^+, \lambda) - \mathcal{L}_{\rho_t}(\xi, \lambda) \leq \tau_t \Delta_P(\xi, z, z^+) - \frac{\tau_t}{2}\|z^+ - z\|_Q^2 - \frac{\sigma}{2}\|\xi - z^+\|^2 - \frac{\delta\rho_t}{2}\|\mathcal{A}z^+ - b\|^2$$

- For any matrix $P \succeq 0$ and any three vectors $u, v, w \in \mathbb{R}^n$:

$$\Delta_P(u, v, w) := \frac{1}{2}\|u - v\|_P^2 - \frac{1}{2}\|u - w\|_P^2.$$

- When $P \equiv I_n$, the identity matrix, we simply write $\Delta_P(u, v, w) \equiv \Delta(u, v, w)$.

A Unified Framework – FLAG

FLAG – **F**aster **LAG**rangian based method

1. **Input:** **Problem data** $[\Psi, \mathcal{A}, b, \sigma]$, and a **nice primal algorithmic map** $\mathrm{Prim}_t(\cdot)$.
2. **Initialization:** Set $t_0 = 1$, $\mu \in (0, \delta]$ and $\rho > 0$. Start with any $\left(x^0, z^0, y^0\right)$.

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2. **Initialization:** Set $t_0 = 1$, $\mu \in (0, \delta]$ and $\rho > 0$. Start with any $\left(x^0, z^0, y^0\right)$.
3. **Iterations:** Generate $\left\{\left(x^k, z^k, y^k\right)\right\}_{k\in\mathbb{N}}$ and $\{t_k\}_{k\in\mathbb{N}}$ via

   3.1. Compute

$$\lambda^k = y^k + \rho_k\left(t_k - 1\right)\left(\mathcal{A}x^k - b\right), \quad \text{with} \quad \rho_k = \begin{cases} \rho, & \text{if } \sigma = 0 \\ \rho t_k & \text{if } \sigma > 0. \end{cases}$$

   3.2. Update the sequence $\left\{\left(x^k, z^k, y^k\right)\right\}_{k\in\mathbb{N}}$ by

$$\begin{aligned} z^{k+1} &\in \mathrm{Prim}_k\left(z^k, \lambda^k\right), \\ y^{k+1} &= y^k + \mu\rho_k\left(\mathcal{A}z^{k+1} - b\right), \\ x^{k+1} &= \left(1 - t_k^{-1}\right)x^k + t_k^{-1}z^{k+1}. \quad \textbf{The acceleration step!} \end{aligned}$$

   3.3. Update the sequence $\{t_k\}_{k\in\mathbb{N}}$ by solving the equation $t_{k+1}^p - t_k^p = t_{k+1}^{p-1}$, *i.e.*,

$$t_{k+1} = \begin{cases} t_k + 1, & p = 1 \ (\textbf{convex case}), \\ \left(1 + \sqrt{1 + 4t_k^2}\right)/2, & p = 2 \ (\textbf{strongly convex case}). \end{cases}$$

FLAG - A Few Comments

- Setting $t_k \equiv 1$ in FLAG, implies $\rho_k \equiv \rho$, $\lambda^k \equiv y^k$, and $x^k \equiv z^k$, thus **recovering the classical basic Lagrangian-based methods**.

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- Borrows ideas from the fundamental work on acceleration of Nesterov (1983).
- The choice of t_k plays a key role in accelerating the nice primal map Prim_t. Both the augmented parameter ρ_k and the prox parameter τ_k are determined and chosen through the recursion which defines the sequence t_k.

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- A main new feature of FLAG is the **auxiliary variable** λ^k defined by:

$$\lambda^k = y^k + \rho_k \left(t_k - 1\right)\left(\mathcal{A}x^k - b\right),$$

which enable us to derive the new faster **non-ergodic** rate of convergence results!

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  which enable us to derive the new faster **non-ergodic** rate of convergence results!
- Prim_t is assumed to be **nice primal algorithmic map** and this is **all we need** to guarantee rate of convergence results (classical and fast)!

The Two Main Pillars of the Analysis

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- Here, it relies on two key lemmas, admitting simple proofs; half-page each!

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Lemma 1

Let $\left\{\left(x^k, z^k, y^k\right)\right\}_{k\in\mathbb{N}}$ generated by FLAG. Then, for any $\xi \in \mathcal{F}$, $\eta \in \mathbb{R}^m$ and $k \geq 0$,

$$\mathcal{L}_{\rho_k}\left(z^{k+1}, \eta\right) - \mathcal{L}_{\rho_k}\left(\xi, \eta\right) \leq \tau_k \Delta_P\left(\xi, z^k, z^{k+1}\right) - \frac{\sigma}{2}\left\|\xi - z^{k+1}\right\|^2 + \frac{1}{\mu\rho_k}\Delta\left(\eta, y^k, y^{k+1}\right)$$
$$- \rho t_{k-1}^p \left\langle \mathcal{A}x^k - b, \mathcal{A}z^{k+1} - b\right\rangle.$$

Proof of Lemma 1

Proof. Since $Prim_k$ is nice, we have

$$\mathcal{L}_{\rho_k}\left(z^{k+1},\lambda^k\right)-\mathcal{L}_{\rho_k}\left(\xi,\lambda^k\right)\leq\tau_k\Delta_P\left(\xi,z^k,z^{k+1}\right)-\frac{\sigma}{2}\left\|\xi-z^{k+1}\right\|^2-\frac{\delta\rho_k}{2}\left\|\mathcal{A}z^{k+1}-b\right\|^2.$$

From the multiplier update and the three-points identity, we obtain, for all $\eta\in\mathbb{R}^m$,

$$\left\langle\eta-y^k,\mathcal{A}z^{k+1}-b\right\rangle=\frac{1}{\mu\rho_k}\left\langle\eta-y^k,y^{k+1}-y^k\right\rangle=\frac{1}{\mu\rho_k}\Delta\left(\eta,y^k,y^{k+1}\right)+\frac{\mu\rho_k}{2}\left\|\mathcal{A}z^{k+1}-b\right\|^2.$$

Using the update rule of the sequence $\{t_k\}_{k\in\mathbb{N}}$ we have that $\rho_k\left(t_k-1\right)=\rho t_{k-1}^p$ and thus

$$\lambda^k=y^k+\rho_k\left(t_k-1\right)\left(\mathcal{A}x^k-b\right)=y^k+\rho t_{k-1}^p\left(\mathcal{A}x^k-b\right),$$

therefore $\left\langle\eta-\lambda^k,\mathcal{A}z^{k+1}-b\right\rangle=\left\langle\eta-y^k,\mathcal{A}z^{k+1}-b\right\rangle-\rho t_{k-1}^p\left\langle\mathcal{A}x^k-b,\mathcal{A}z^{k+1}-b\right\rangle.$

Combining these relations yields (recall that $\mu\leq\delta$) the desired result. □

Lemma 2

Let $\left\{\left(x^k, z^k, y^k\right)\right\}_{k\in\mathbb{N}}$ generated by FLAG. Then, for any $\xi \in \mathcal{F}$, $\eta \in \mathbb{R}^m$ and $k \geq 0$,

$$t_k^p s_{k+1} - t_{k-1}^p s_k \leq \frac{\tau_k \rho_k}{\rho} \Delta_P\left(\xi, z^k, z^{k+1}\right) - \frac{\rho_k \sigma}{2\rho}\left\|\xi - z^{k+1}\right\|^2 + \frac{1}{\mu\rho}\Delta\left(\eta, y^k, y^{k+1}\right),$$

where $s_k = \mathcal{L}_{\rho t_{k-1}^p}\left(x^k, \eta\right) - \mathcal{L}_{\rho t_{k-1}^p}\left(\xi, \eta\right)$.

Proof of Lemma 2

Proof. Since $x^{k+1} = \left(1 - t_k^{-1}\right) x^k + t_k^{-1} z^{k+1}$, we obtain from the convexity of $\Psi(\cdot)$ that

$$\Psi\left(x^{k+1}\right) \leq \left(1 - t_k^{-1}\right) \Psi\left(x^k\right) + t_k^{-1} \Psi\left(z^{k+1}\right).$$

Therefore, multiplying both sides by t_k^p (recalling that $t_k^p - t_k^{p-1} = t_{k-1}^p$), we obtain

$$t_k^p \left(\Psi\left(x^{k+1}\right) - \Psi(\xi)\right) - t_{k-1}^p \left(\Psi\left(x^k\right) - \Psi(\xi)\right) \leq t_k^{p-1} \left(\Psi\left(z^{k+1}\right) - \Psi(\xi)\right).$$

We also have that

$$t_k^p \left\langle \eta, \mathcal{A}x^{k+1} - b \right\rangle - t_{k-1}^p \left\langle \eta, \mathcal{A}x^k - b \right\rangle = t_k^{p-1} \left\langle \eta, \mathcal{A}z^{k+1} - b \right\rangle.$$

Combining these two facts yields

$$t_k^p \left(\mathcal{L}\left(x^{k+1}, \eta\right) - \mathcal{L}(\xi, \eta)\right) - t_{k-1}^p \left(\mathcal{L}\left(x^k, \eta\right) - \mathcal{L}(\xi, \eta)\right) \leq t_k^{p-1} \left(\mathcal{L}\left(z^{k+1}, \eta\right) - \mathcal{L}(\xi, \eta)\right).$$

Now, we will again use that $x^{k+1} = \left(1 - t_k^{-1}\right) x^k + t_k^{-1} z^{k+1}$ to obtain

$$\left\|\mathcal{A}x^{k+1} - b\right\|^2 = \left(1 - t_k^{-1}\right)^2 \left\|\mathcal{A}x^k - b\right\|^2 + t_k^{-2} \left\|\mathcal{A}z^{k+1} - b\right\|^2 + 2\left(1 - t_k^{-1}\right) t_k^{-1} \left\langle \mathcal{A}x^k - b, \mathcal{A}z^{k+1} - b \right\rangle.$$

Multiplying both sides of the above equality by $\rho t_k^{2p}/2$ yields

$$\frac{\rho t_k^{2p}}{2} \left\|\mathcal{A}x^{k+1} - b\right\|^2 - \frac{\rho t_{k-1}^{2p}}{2} \left\|\mathcal{A}x^k - b\right\|^2 = \frac{\rho_k t_k^{p-1}}{2} \left\|\mathcal{A}z^{k+1} - b\right\|^2 + \rho_k t_{k-1}^p \left\langle \mathcal{A}x^k - b, \mathcal{A}z^{k+1} - b \right\rangle.$$

Therefore, with $s_k = \mathcal{L}_{\rho t_{k-1}^p}\left(x^k, \eta\right) - \mathcal{L}_{\rho t_{k-1}^p}(\xi, \eta)$, we have

$$t_k^p s_{k+1} - t_{k-1}^p s_k \leq t_k^{p-1} \left(\mathcal{L}_{\rho_k}\left(z^{k+1}, \eta\right) - \mathcal{L}_{\rho_k}(\xi, \eta)\right) + \rho_k t_{k-1}^p \left\langle \mathcal{A}x^k - b, \mathcal{A}z^{k+1} - b \right\rangle.$$

Proof of Lemma 2 – End

From the previous lemma, after we multiplied both sides by t_k^{p-1} (recall that $\rho_k = \rho t_k^{p-1}$), we have

$$t_k^{p-1}\left(\mathcal{L}_{\rho_k}\left(z^{k+1},\eta\right)-\mathcal{L}_{\rho_k}\left(\xi,\eta\right)\right)+\rho_k t_{k-1}^{p}\left\langle \mathcal{A}x^k-b,\mathcal{A}z^{k+1}-b\right\rangle \leq$$

$$\frac{\tau_k\rho_k}{\rho}\Delta_P\left(\xi,z^k,z^{k+1}\right)-\frac{\rho_k\sigma}{2\rho}\left\|\xi-z^{k+1}\right\|^2+\frac{1}{\mu\rho}\Delta\left(\eta,y^k,y^{k+1}\right).$$

By combining the last two inequalities, we obtain the desired result. □

Types of Rate of Convergence – Many Results

We focus on non-asymptotic rate of convergence (iteration complexity) using the following two classical measures:

(i) **Function values gap** in terms of $\Psi\left(x^k\right) - \Psi\left(x^*\right)$.

(ii) **Feasibility violation** of the constraints of problem (P) in terms of $\left\|\mathcal{A}x^k - b\right\|$.

Other measures in the literature: Lagrangian, $\left\|x^k - x^*\right\|^2$, $\left\|x^{k+1} - x^k\right\|^2$, etc.

When discussing these measures, we also distinguish between rates expressed in terms of the **original produced sequence** or of the **ergodic sequence**.

Types of Rate of Convergence – Many Results

We focus on non-asymptotic rate of convergence (iteration complexity) using the following two classical measures:

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- Non-ergodic $O(1/N)$ rate of convergence result in terms of function values and feasibility violation for the **specific Linearized ADMM** (Li and Lin (19)).

Main Results - Non-Ergodic Rate: $O(1/N^2)$ and $O(1/N)$

The strongly convex case $\sigma > 0$.

Theorem 1. (A fast non-ergodic function values and feasibility violation rates)

Let $\left\{\left(x^k, z^k, y^k\right)\right\}_{k\in\mathbb{N}}$ be a sequence generated by FLAG. **Suppose that $\sigma > 0$ and** $0 \preceq P \preceq (\sigma/2)\, I_n$. Then, for any optimal solution x^* of problem (P) and $N \geq 1$,

$$\Psi\left(x^N\right) - \Psi\left(x^*\right) \leq \frac{B_{\rho,c}\left(x^*\right)}{2N^2} \qquad \text{and} \qquad \left\|\mathcal{A}x^N - b\right\| \leq \frac{B_{\rho,c}\left(x^*\right)}{cN^2},$$

where $B_{\rho,c}\left(x^*\right) := 4\left(\left\|x^* - z^0\right\|_P^2 + \frac{1}{\mu\rho}\left(\left\|y^0\right\| + c\right)^2\right)$.

Main Results - Non-Ergodic Rate: $O(1/N^2)$ and $O(1/N)$

The convex case $\sigma = 0$.

Theorem 2. (A non-ergodic function values and feasibility violation rates)

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Ergodic type results

FLAG is versatile

- Deriving weaker results of ergodic type was not our primary goal.
- Nevertheless, our main framework FLAG easily adapt to that task. (The sequences $\left\{x^k\right\}_{k\in\mathbb{N}}$ and $\left\{\lambda^k\right\}_{k\in\mathbb{N}}$ are not used for that scenario!)

See details in the paper.

Ergodic Rate of Convergence Results

Corollary 1. (A fast ergodic function values and feasibility violation rates)

Let $\left\{\left(z^k, y^k\right)\right\}_{k\in\mathbb{N}}$ be a sequence generated by FLAG. **Suppose that** $\sigma > 0$ **and** $0 \preceq P \preceq (\sigma/2)\, I_n$. Then, for any optimal solution z^* of problem (P), the following holds for the **ergodic sequence** $\bar{z}^N = t_{N-1}^{-2} \sum_{k=0}^{N-1} t_k z^{k+1}$

$$\Psi\left(\bar{z}^N\right) - \Psi\left(z^*\right) \leq \frac{B_{\rho,c}\left(z^*\right)}{2N^2} \qquad \text{and} \qquad \left\|\mathcal{A}\bar{z}^N - b\right\| \leq \frac{B_{\rho,c}\left(z^*\right)}{cN^2},$$

where $B_{\rho,c}\left(z^*\right) := 4\left(\left\|z^* - z^0\right\|_P^2 + \frac{1}{\mu\rho}\left(\left\|y^0\right\| + c\right)^2\right)$.

Corollary 2. (An ergodic function values and feasibility violation rates)

Let $\left\{\left(z^k, y^k\right)\right\}_{k\in\mathbb{N}}$ be a sequence generated by FLAG with $\sigma = 0$ **and** $t_k = 1$ **for all** $k \in \mathbb{N}$. Then, for any optimal solution z^* of problem (P), the following holds for **the ergodic sequence** $\bar{z}^N = N^{-1} \sum_{k=0}^{N-1} z^{k+1}$

$$\Psi\left(\bar{z}^N\right) - \Psi\left(z^*\right) \leq \frac{B_{\rho,c}\left(z^*\right)}{2N} \qquad \text{and} \qquad \left\|\mathcal{A}\bar{z}^N - b\right\| \leq \frac{B_{\rho,c}\left(z^*\right)}{cN},$$

where $B_{\rho,c}\left(z^*\right) := 2\left(\left\|z^* - x^0\right\|_P^2 + \frac{1}{\mu\rho}\left(\left\|y^0\right\| + c\right)^2\right)$.

Nice Primal Algorithmic Map for Block Model

The notion of nice algorithmic map is flexible and **easily adapt to the block setting:**

$$\min_{x:=(u,v)\in\mathbb{R}^p\times\mathbb{R}^q=\mathbb{R}^n}\left\{\Psi(x):=f(u)+g(v)\ :\ \mathcal{A}x:=Au+Bv=b\right\}.$$

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Definition (Nice primal algorithmic map - Block version)

Given the parameters $\rho, t>0$, we let $(\rho_t,\tau_t)=(\rho,t^{-1})$ (when $\sigma=0$) and $(\rho_t,\tau_t=(\rho t,t)$ (when $\sigma>0$). A primal algorithmic map $\mathrm{Prim}_t:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$, which is applied on the augmented Lagrangian $\mathcal{L}_{\rho_t}(z,\lambda)$, that generates $z^+=(u^+,v^+)$ via $z^+\in\mathrm{Prim}_t(z,\lambda)$, is *nice*, if there exist $\delta\in(0,1]$ and $P_1,Q_1\in\mathbb{S}^p_+$ and $P_2,Q_2\in\mathbb{S}^q_+$ with $P=(P_1,P_2)$ and $Q=(Q_1,Q_2)$, s.t. for any $(\xi_1,\xi_2)\in\mathcal{F}$

$$\mathcal{L}_{\rho_t}\left(z^+,\lambda\right)-\mathcal{L}_{\rho_t}(\xi,\lambda)\le\frac{1}{t}\Delta_{P_1}\left(\xi_1,u,u^+\right)-\frac{1}{2t}\left\|u^+-u\right\|^2_{Q_1}+\tau_t\Delta_{P_2}\left(\xi_2,v,v^+\right)$$
$$-\frac{\tau_t}{2}\left\|v^+-v\right\|^2_{Q_2}-\frac{\sigma}{2}\left\|\xi_2-v^+\right\|^2-\frac{\delta\rho_t}{2}\left\|\mathcal{A}z^+-b\right\|^2.$$

• **Note:** Here we use g strongly convex. Hence, only the block v uses τ_t (to cover both convex/strongly convex cases). For the other block u (with only convexity. i.e., $\sigma=0$), we fixed $\tau_t=t^{-1}$.

Iconic Lagrangian-based Methods Admit a Nice Primal Algorithmic Map!

- Augmented Lagrangian Methods (classical, proximal, and prox-linearized)
- Alternating Direction Method of Multipliers ADMM
- Proximal ADMM
- Proximal Linearized ADMM
- Chambolle-Pock Method
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For each method an explicit parameter δ and matrices P, Q can be found!
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Meaning, they all **admit Nice Primal Algorithmic Map!**

Therefore, our nonergodic convergence rate results can be applied.

In addition, nice primal algorithmic maps, can be also be identified for problems with **composite objective**...

Model (P) with the Sum Composite Objective Function

We consider the sum composite model: nonsmooth + smooth objective

$$\min_{x \in \mathbb{R}^n} \{f(x) + h(x) \ : \ \mathcal{A}x = b\},$$

- $f : \mathbb{R}^n \to (-\infty, +\infty]$ is a proper, lsc and σ-strongly convex ($\sigma \geq 0$),
- $h : \mathbb{R}^n \to \mathbb{R}$ is convex C^1 with **L-Lipschitz continuous gradient**.

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Lemma (Proximal AL is nice)

Let $M \succeq LI_n$, the primal algorithmic map $\mathrm{Prim}_t(\cdot)$ *defined by*

$$z^+ = \mathrm{argmin}_\xi \left\{ f(\xi) + \langle \nabla h(z), \xi \rangle + \langle \lambda, \mathcal{A}\xi - b \rangle + \frac{\rho_t}{2}\|\mathcal{A}\xi - b\|^2 + \frac{\tau_t}{2}\|\xi - z\|_M^2 \right\},$$

is nice with $\delta = 1$ and $P = M$ and $Q = M - LI_n$.

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Lemma (Proximal Linearized AL is nice)

Let $M \succeq \rho\mathcal{A}^T\mathcal{A} + LI_n$, the primal algorithmic map $\mathrm{Prim}_t(\cdot)$ defined by

$$z^+ = \mathrm{argmin}_\xi \left\{ f(\xi) + \langle \nabla h(z), \xi \rangle + \langle \lambda, \mathcal{A}\xi - b \rangle + \rho_t \langle \mathcal{A}z - b, \mathcal{A}\xi \rangle + \frac{\tau_t}{2}\|\xi - z\|_M^2 \right\},$$

is nice with $\delta = 1$ and $P = M - \rho\mathcal{A}^T\mathcal{A}$ and $Q = M - \rho\mathcal{A}^T\mathcal{A} - LI_n$.

Epilogue

A Simple Recipe for Rate of Convergence of Lagrangian-based Methods

(i) Formulate the problem at hand via model (P), *i.e.*, **identify the relevant problem data** $[\Psi, \mathcal{A}, b, \sigma]$. **The value of σ will determine the type of rate that can be achieved (classical or fast)**.

(ii) **Define the desired iterative step(s) of the primal algorithmic map** $\mathrm{Prim}_t(\cdot)$ applied on the augmented Lagrangian $\mathcal{L}_{\rho_t}(\cdot)$ of model (P).

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(iv) Apply Theorem 1 (if $\sigma > 0$) or Theorem 2 (if $\sigma = 0$) for the corresponding FLAG to **obtain a faster non-ergodic rate of convergence for the designed method**.

Therefore, there is no need any more to enter into the machinery of the proofs!

Sabach, S. and Teboulle, M. Faster Lagrangian-Based Methods in Convex Optimization (November, 2020).

Thank you for listening!

Happy Birthday Yurii !

http://www.math.tau.ac.il/~teboulle/